EXERCISES

1. The Main Theorem of Galois Theory

1. Determine the irreducible polynomial for $i + \sqrt{2}$ over $\mathbb{Q}$.
2. Prove that the set $(1, i, \sqrt{2}, i\sqrt{2})$ is a basis for $\mathbb{Q}(i, \sqrt{2})$ over $\mathbb{Q}$.
3. Determine the intermediate fields between $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.
4. Determine the intermediate fields of an arbitrary biquadratic extension without appealing to the Main Theorem.
5. Prove that the automorphism $\mathbb{Q}(\sqrt{2})$ sending $\sqrt{2}$ to $-\sqrt{2}$ is discontinuous.
6. Determine the degree of the splitting field of the following polynomials over $\mathbb{Q}$.
   (a) $x^4 - 1$ **(b)** $x^3 - 2$ **(c)** $x^4 + 1$
7. Let α denote the positive real fourth root of 2. Factor the polynomial $x^4 - 2$ into irreducible factors over each of the fields $\mathbb{Q}$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{2}, i)$, $\mathbb{Q}(\alpha)$, $\mathbb{Q}(\alpha, i)$.
8. Let $\zeta = e^{2\pi i/5}$.
   (a) Prove that $K = \mathbb{Q}(\zeta)$ is a splitting field for the polynomial $x^5 - 1$ over $\mathbb{Q}$, and determine the degree $[K : \mathbb{Q}]$.
   (b) Without using Theorem (1.11), prove that K is a Galois extension of $\mathbb{Q}$, and determine its Galois group.
9. Let K be a quadratic extension of the form $F(\alpha)$, where $\alpha^2 = a \in F$. Determine all elements of K whose squares are in F.
10. Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$. Determine $[K : \mathbb{Q}]$, prove that K is a Galois extension of $\mathbb{Q}$, and determine its Galois group.
11. Let K be the splitting field over $\mathbb{Q}$ of the polynomial $f(x) = (x^2 - 2x - 1)(x^2 - 2x - 7)$. Determine $G(K/\mathbb{Q})$, and determine all intermediate fields explicitly.
12. Determine all automorphisms of the field $\mathbb{Q}(\sqrt[3]{2})$.
13. Let K/F be a finite extension. Prove that the Galois group $G(K/F)$ is a finite group.
14. Determine all the quadratic number fields $\mathbb{Q}[\sqrt{d}]$ which contain a primitive pth root of unity, for some prime $p \neq 2$.
15. Prove that every Galois extension K/F whose Galois group is the Klein four group is biquadratic.
16. Prove or disprove: Let $f(x)$ be an irreducible cubic polynomial in $\mathbb{Q}[x]$ with one real root α. The other roots form a complex conjugate pair $\beta, \overline{\beta}$, so the field $L = \mathbb{Q}(\beta)$ has an automorphism σ which interchanges $\beta, \overline{\beta}$.
17. Let K be a Galois extension of a field F such that $G(K/F) \approx C_2 \times C_{12}$. How many intermediate fields L are there such that **(a)** $[L : F] = 4$, **(b)** $[L : F] = 9$, **(c)** $G(K/L) \approx C_4$?
18. Let $f(x) = x^4 + bx^2 + c \in F[x]$, and let K be the splitting field of f. Prove that $G(K/F)$ is contained in a dihedral group D_4.
19. *Let $F = \mathbb{F}_2(u)$ be the rational function field over the field of two elements. Prove that the polynomial $x^2 - u$ is irreducible in $F[x]$ and that it has two equal roots in a splitting field.

* **20.** Let F be a field of characteristic 2, and let K be an extension of F of degree 2.
  (a) Prove that K has the form $F(\alpha)$, where α is the root of an irreducible polynomial over F of the form $x^2 + x + a$, and that the other root of this equation is $\alpha + 1$.
  (b) Is it true that there is an automorphism of K sending $\alpha \rightsquigarrow \alpha + 1$?

2. Cubic Equations

1. Prove that the discriminant of a real cubic is positive if all the roots are real, and negative if not.

2. Determine the Galois groups of the following polynomials.
(a) $x^3 - 2$ **(b)** $x^3 + 27x - 4$ **(c)** $x^3 + x + 1$ **(d)** $x^3 + 3x + 14$
(e) $x^3 - 3x^2 + 1$ **(f)** $x^3 - 21x + 7$ **(g)** $x^3 + x^2 - 2x - 1$
(h) $x^3 + x^2 - 2x + 1$

3. Let f be an irreducible cubic polynomial over F, and let δ be the square root of the discriminant of f. Prove that f remains irreducible over the field $F(\delta)$.

4. Let α be a complex root of the polynomial $x^3 + x + 1$ over $\mathbb{Q}$, and let K be a splitting field of this polynomial over $\mathbb{Q}$.
(a) Is $\sqrt{-3}$ in the field $\mathbb{Q}(\alpha)$? Is it in K?
(b) Prove that the field $\mathbb{Q}(\alpha)$ has no automorphism except the identity.

***5.** Prove Proposition (2.16) directly for a cubic of the form (2.3), by determining the formula which expresses α_2 in terms of α_1, δ, p, q explicitly.

6. Let $f \in \mathbb{Q}[x]$ be an irreducible cubic polynomial which has exactly one real root, and let K be its splitting field over $\mathbb{Q}$. Prove that $[K : \mathbb{Q}] = 6$.

7. When does the polynomial $x^3 + px + q$ have a multiple root?

8. Determine the coefficients p, q which are obtained from the general cubic (2.1) by the substitution (2.2).

9. Prove that the discriminant of the cubic $x^3 + px + q$ is $-4p^3 - 27q^2$.

3. Symmetric Functions

1. Derive the expression (3.10) for the discriminant of a cubic by the method of undetermined coefficients.

2. Let $f(u)$ be a symmetric polynomial of degree d in $u_1, \ldots, u_n$, and let $f^0(u_1, \ldots, u_{n-1}) = f(u_1, \ldots, u_{n-1}, 0)$. Say that $f^0(u) = g(s^0)$, where s_i^0 are the elementary symmetric functions in $u_1, \ldots, u_{n-1}$. Prove that if $n > d$, then $f(u) = g(s)$.

3. Compute the discriminant of a quintic polynomial of the form $x^5 + ax + b$.

4. With each of the following polynomials, determine whether or not it is a symmetric function, and if so, write it in terms of the elementary symmetric functions.
(a) $u_1^2u_2 + u_2^2u_1$ $(n = 2)$
(b) $u_1^2u_2 + u_2^2u_3 + u_3^2u_1$ $(n = 3)$
(c) $(u_1 + u_2)(u_2 + u_3)(u_1 + u_3)$ $(n = 3)$
(d) $u_1^3u_2 + u_2^3u_3 + u_3^3u_1 - u_1u_2^3 - u_2u_3^3 - u_3u_1^3$ $(n = 3)$
(e) $u_1^3 + u_2^3 + \cdots + u_n^3$

5. Find two natural bases for the ring of symmetric functions, as free module over the ring R.